The Effect of Geometry on Pointwise Decay of Waves

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$$-\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0$$

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2. Wave Equation

$$-\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

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Well, no.

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Well, no.

- not answering a specific scientific question
- exploring mathematical structure of equations
- proof focused

What's the Point of Pure Math Research?

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Example:

Abstract geometry developed by Gauss, Riemann, and others during the 19th century:

- non-Euclidean
- defining geometry without considering a background space
- geometry determined by how distances are measured on a surface

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These ideas are built off of decades of work of countless mathematicians.

General Relativity: The Basics

1. Space-time

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Known Space-times

Schwarzschild: stationary black hole

Kerr: rotating black hole

Price's Law

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Solutions to the wave equation decay like t^{-3} on Schwarzshild

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Proved in Tataru 2013:

t^{-3} decay rate for solutions on space-times that

1. Tend toward flat geometry at a rate of r^{-1}

   i.e. as $r \to \infty$, measure of distance $\to$ Euclidean

2. Satisfy weak local energy decay
   - in general, energy is conserved
   - enough "spreading out" $\Rightarrow$ energy decay on compact sets

More on Weak Local Energy Decay

Local Energy Decay Estimates have a long history (including Jason and his advisor)

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They are sensitive to...

1. Trapping
2. Eigenvalues/Resonances

My Research

What happens to pointwise decay if the rate towards Euclidean geometry is changed?

Encoding Geometric Assumptions

1. Rate towards flat metric $\mapsto$ definition of operator

$$P = -\partial_t^2 + \Delta + P^1 \partial_t + P^2$$

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2. Weak local energy decay $\mapsto$ integral bounds on Fourier transform of solution

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Take Fourier transform of Pu and integrate by parts to find

$$P_\tau \hat{u}(\tau) = g$$

$$P_\tau = -\tau^2 - i\tau P^1 + \Delta + P^2$$

Fourier Transform

Relating the Fourier transform of a solution to the initial data is useful because

1. The Fourier transform preserves "L^2 norm" (i.e. integral of the square of a function)
2. It is invertible - can recover original function

$$u(t, x) = \int_{-\infty}^{\infty} e^{it\tau} \hat{u}(\tau, x) d\tau$$

Define $R_\tau g = \hat{u}(\tau)$

$$R_\tau = P_\tau^{-1}$$

Preliminary Results and Future Questions

It is expected that faster tendency toward flat will result in faster decay.

Flatness Rate $\to$ Operator $\to$ Expression for R_0 $\to$ Low frequency bounds on $\hat{u}$

Weak local energy decay does not measure rate toward flat metric.

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Flatness Rate $\rightarrow$ Operator $\rightarrow$ Expression for R_0 $\rightarrow$ Low frequency bounds on $\hat{u}$

Weak local energy decay does not measure rate toward flat metric.

What does weak local energy decay measure?

1. Trapping
2. Resonances
   - But what does this say about the *geometry*?

Thank You